Fokker Planck and Master equation

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Connections

- Temporal evolution of the probability
- Hypotesis for the Chapman-Kolmogorov equation
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- The master equation is equivalent to a stochastic differential equation
- Derivation of the Fokker-Plank from a Langevin equation

Transition Probability

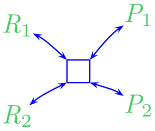
Transition probability at a time τ between two states $y_3 \to y_2$ in a τ exspansion

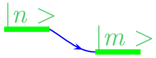
$$\mathcal{P}_\tau(y_3|y_2) = \delta(y_3 - y_2) + W(y_3|y_2)\tau + O(\tau^2)$$

$W(y_3|y_2) = \mathrm{d}_t\mathcal{P}(y_3|y_2)$ transition rate
Normalizing: $\alpha(y_2) = \int W(y_3|y_2)\mathrm{d}y_3$

$$\begin{aligned}\mathcal{P}_\tau(y_3|y_2) &= (1 - \tau\alpha(y_2))\delta(y_3 - y_2) + W(y_3|y_2)\tau \\ &= \delta(y_3 - y_2) + W(y_3|y_2)\tau - W(y_2|y_3)\tau + O(\tau^2)\end{aligned}$$

A Chemical reaction: $R_1 + R_2 \rightleftharpoons P_1 + P_2$ Fermi's Rule: $\lambda_{if} = \frac{2\pi}{\hbar}|M_{if}|^2\rho_f$





Chapman-Kolmogorow equation

The probability of a Markov process depends only on the probability of the last process

$$\mathcal{P}(y_n|y_{n-1},,y_1) = \mathcal{P}(y_n|y_{n-1})$$

We assume a stationary ($\mathcal{P}(t) = \mathcal{P}(t+\tau)$) and homogenues process ($\mathcal{P}(t_1, t_2) = \mathcal{P}(t_1 - t_2)$) (a more general case is the Boltzmann equation)
The conditional probability between the state y_3 and y_1 can be written:

$$\mathcal{P}(y_3|y_1) = \int \mathcal{P}(y_3|y_2)\mathcal{P}(y_2|y_1)\mathrm{d}y_2$$

One step process: Brownian motion, shot noise, dacay.

Continous master equation

We derivate the conditional probability with respect to the first order
$\partial_\tau \mathcal{P}(y_3|y_2) = W(y_3|y_2) - W(y_2|y_3)$
The derivative of the conditional probability is:

$$\partial_\tau \mathcal{P}(y_3|y_1) = \int \Big(W(y_3|y_2)\mathcal{P}(y_2|y_1) - W(y_2|y_3)\mathcal{P}(y_3|y_1) \Big) \mathrm{d}y_2$$

We remove the stationarity condition, multiply by $P(y_1, t)$ and integrate over x_1

$$\mathrm{d}_t \mathcal{P}(y, t) = \int W(y, y')\mathcal{P}(y', t) - W(y', y)\mathcal{P}(y, t)\mathrm{d}y'$$

Einstein coefficients: Spontaneus emission A_{mn}, Absorption: $B_{nm}J$, Stimulated emission: $B_{mn}J$

Discrete master equation

A chemical reaction generation/ricombination (discrete, nonlinear) $X \underset{W^\dagger}{\overset{W}{\rightleftharpoons}} 2X$:

$$\dot{\mathcal{P}}_n = W^\dagger n(n+1)\mathcal{P}_{n+1} + W(n-1)\mathcal{P}_{n-1} - W^\dagger n\mathcal{P}_n - Wn(n-1)\mathcal{P}_n$$

The master equation says that the probability of a transition in a time t is the sum of the gain in changing from $m \to n$ minus the loss between $n \to m$

$$(n-1) \xrightarrow{W} (n) \xrightarrow{W^\dagger} (n+1)$$

In the steady state the lhs of the master equation is zero $\sum_m W_{nm}\mathcal{P}_m = (\sum_m W_{mn})\mathcal{P}_n$
Detailed balance: $W_{nm}\mathcal{P}_m = W_{mn}\mathcal{P}_n$
Detailed balance is necessary but not sufficient for the equilibrium (microscopic reversibility)

Fokker-Planck equation

The evolution of a single event $\mathcal{P}(t)$ is governed by the master equation which describes all statistical properties.
We recast the integro equation of a master equation into the form of a Kramers-Moyal expansion ($r = x - x'$, small)

$$\begin{aligned} \partial_t \mathcal{P}(x,t) &= \mathcal{P}(x,t) \int W(x|r)\mathcal{P}(x,t) - W(x|-r)\mathcal{P}(x,t)\mathrm{d}r \\ &- \int r\partial_x (W(x|r)\mathcal{P}(x,t))\mathrm{d}r + \frac{1}{2}\int r^2 \partial_x^2 (W(x|r)\mathcal{P}(x,t))\mathrm{d}r \pm \ldots \end{aligned}$$

or

$$\dot{\mathcal{P}}_t(x) = \sum_n \frac{-^n}{n!} \partial_x^n D_{KM}^{(n)} \mathcal{P}_t(x) \quad D_{KM}^{(n)}(x,t) = \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau} < (x(t+\tau) - x(t))^n >$$

From the Pawula Theorem one can either use the first moment, the first and the second or every of them
In Most of the case two moments are enough, (e.g. Gaussian noise):
Fokker-Planck equation

$$\dot{\mathcal{P}}(x,t) = -\partial_x D^{(1)} \mathcal{P}(x,t) + \partial_x^2 D^{(2)} \mathcal{P}(x,t)$$

From master equation to stochastic differential equation

We consider the equation:

$$\dot{x}(t) = X(x) + \xi(t)$$

The moments of the noise ξ are connected with the Kramers-Moyal expansion

$$lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\tau}^{\tau\Delta} \mathrm{d}s < \xi(s)|x(t) = x_0 >= D_{KM}^{(1)} - X(x)$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\tau}^{\tau+\Delta} \mathrm{d}t_1 \ldots \mathrm{d}t_n < \xi(t_1) \ldots \xi(t_n)|x(t) = x_0 >= D_{KM}^{(n)}$$

The first moment is deterministic, the higher oder stochastic (mesoscopic description)

Considering the same moments the following SDEq is equivalent to the previous MEq

A non zero mean of the noise contributes to a drift term

P. Hanggi Z., Physik B, **43**, *269-273,(1981)*

Flux

From the Fokker-Planck equation we define a flux

$$J := -D^{(1)}X(x)\mathcal{P}(x) + \frac{1}{2}D^{(2)}\mathcal{P}(x)$$

The FPEq can be written as a conservation law:

$$\partial_t \mathcal{P}(x) - \partial_x J(x) = 0$$

Tipical boundary condition

- Natural (decay): $\int \mathcal{P}(x) = \mathrm{norm}$
- Reflecting (wall): $J(x = a, t) = 0$
- Absorbing (first passage time): $P(x = a, t) = 0$

Neuron dynamics, role of noise

The dynamics of the voltage in a neuron is ruled by the equation:

$$\dot{V}(t) = -\frac{V(t)}{\tau} + I(t) \qquad I(t) = \mu + \sigma_w \eta(t) + \sigma_w \frac{\beta}{\sqrt{2\tau_c}} z(t)$$

Where $\eta(t)$ is a white noise and $z(t)$ is an auxiliary colored noise, V is a potential between H and Θ

$$C(t,t') = <(I(t) - <I(t)>)(I(t') - <I(t')>)> = \sigma_w^2 \delta(t-t') + \frac{\Sigma_2}{2\tau_c} e^{-\frac{|t-t'|}{\tau_c}}$$

The Fokker-Planck equation is:

$$\left(\partial_V \left(f(V) - \mu + \frac{\sigma^2}{2} \partial_V \right) + \frac{1}{\tau_c} \partial_z (z + \partial_z) - \sqrt{\frac{2\sigma^2 \alpha^2}{\tau_c}} \partial_V \right) \mathcal{P} = -\delta(V-H) J(z)$$

The *firing rate* ν is the probability per unit time that the potential cross a threshold Θ. $J(z)$ is the escape probability current.

Firing rate

$\mathcal{P}(V, z)$ is the staedy state probability. We suppose $\tau_c < \tau_{ref}$ (correlation, refractory).

$$J(z) = \frac{\nu_{out}}{\sqrt{2\pi}} e^{-x^2/2}$$

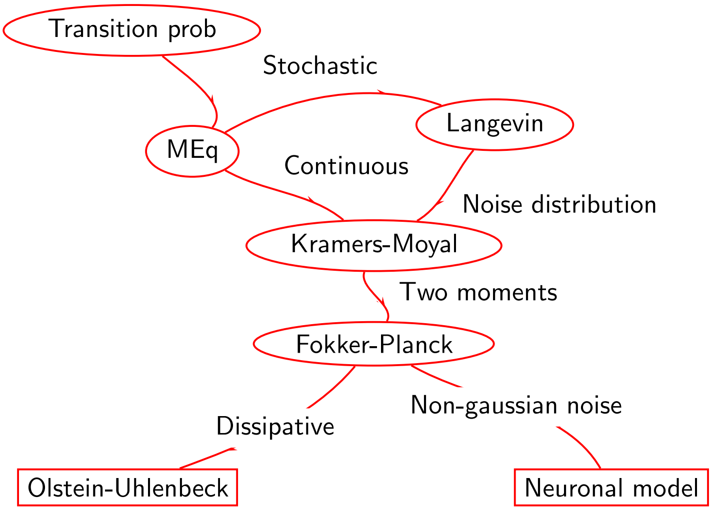
z after a spike relaxes to the stationary distribution. The Fokker-Planck has to be resolved with the normalisation condition

$$\nu_{out}\tau_{ref} + \int_{-\infty}^{\Theta} \mathrm{d}V \int_{-\infty}^{\infty} \mathrm{d}z \mathcal{P}(V, z) = 1$$

Finally the output firing rate is give by

$$\nu_{out} = \int \mathrm{d}z J(z)$$

Summary


Transition prob
Stochastic
Langevin
MEq
Continuous
Noise distribution
Kramers-Moyal
Two moments
Fokker-Planck
Dissipative
Non-gaussian noise
Olstein-Uhlenbeck
Neuronal model

Thank you for your attention

Aknowledgments
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References

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R. Moreno, J. de la Rocha, A. Renart, N. Parga
PRL **89** 288101 (2002)

How Spike Generation Mechanisms Determine the Neuronal Response to Fluctuating Inputs
N. Foucard-Trocmé, D. Hansel, C. van Vreeswijk, N. Brunel
J. Neuroscience **23** 11628

Langevin equation

$$\dot{x} = X(x) + \xi(t)$$

The probability distribution $P(y,t)$ is defined as:

$$P(y,t) =< \delta(y - x(t)) >_\xi$$

Advection (drift) term $D^{(1)}$: $\dot{x} = X(x)$

$$\begin{aligned}\partial_t \mathcal{P}(y,t) &= -\dot{x}\mathrm{d}_y\delta(y-x) = -X(x)\mathrm{d}_y\delta(y-x) = -\mathrm{d}_y(\delta(y-x)F(x)) \\ &= -\mathrm{d}_y(\delta(y-x)F(y)) = -\mathrm{d}_y(F(y)\mathcal{P}(y,t))\end{aligned}$$

Diffusive term $D^{(2)}$: $\dot{x} = \xi(t)$

$$\mathcal{P}(x,t) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{y^2}{2\Delta t}} \qquad \partial_t\mathcal{P}(x,t) = \frac{D^{(2)}}{2}\partial_x^2\mathcal{P}(x,t)$$

$$\partial_t\mathcal{P}(x,t) = -D^{(1)}\partial_x\mathcal{P}(x,t) + D^{(2)}\partial_x^2\mathcal{P}(x,t)$$

Ornstein-Uhlenbeck process

We start from the dissipative Langevin equation

$$\Delta v(t) = -\gamma v(t) + \sigma \Delta \xi(t)$$

The KM coefficients are $D^{(1)} = -\gamma v$, $D^{(2)} = \frac{\sigma^2}{2}$
The stationary solution is given by $\dot{\mathcal{P}}(x) \stackrel{!}{=} 0$

$$\mathcal{P}(v) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{v^2 m}{2K_B T}}$$